Adaptive in Time Approximation of Parametric Parabolic PDEs

Benjamin Kent (University of Manchester, UK)
with Catherine Powell, David Silvester (University of Manchester, UK),
Małgorzata J. Zimoń (IBM Research UK / University of Manchester, UK)

benjamin.kent@manchester.ac.uk

Motivation

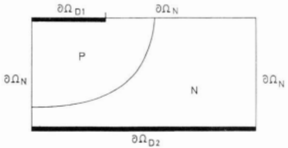


Fig. 3.1.1 *P-N* diode

Markowich, Ringhofer, and Schmeiser 1990

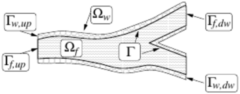


FIG. 2.1. *Computational domain representing a 2D section of a vascular district featuring the lumen Ω_f and the wall Ω_w.*

Quarteroni, Veneziani, and Zunino 2002

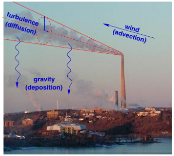


Fig. 1.1 *A photograph of emissions from the Inco Superstack (in Sudbury, ON, Canada) that illustrates the three main contributions to atmospheric contaminant transport: advection from the wind; diffusion from turbulent eddy motion; and deposition owing to gravitational settling.*

Stockie 2011

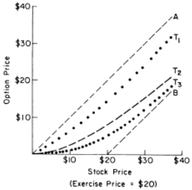


FIG. 1.—The relation between option value and stock price

Black and Scholes 1973

Problem statement

Approximate solution of

$$\frac{\partial u(\boldsymbol{x}, t, \boldsymbol{y})}{\partial t} + A(\boldsymbol{x}, \boldsymbol{y})u(\boldsymbol{x}, t, \boldsymbol{y}) = 0 \quad \rho\text{-a.s. in } \Gamma$$

(+ initial and boundary conditions) for elliptic differential operator $A(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{y} \in \Gamma \subset \mathbb{R}^d$.

$$\boxed{u^{h,l,\delta}(\boldsymbol{x}, t, \boldsymbol{y}) := \sum_{\boldsymbol{z} \in \mathcal{Z}^l} u^{h,\delta}(\boldsymbol{x}, t; \boldsymbol{z}) L_{\boldsymbol{z}}^l(\boldsymbol{y})}$$

Problem statement

Approximate solution of

$$\frac{\partial u(\boldsymbol{x}, t, \boldsymbol{y})}{\partial t} + A(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{x}, t, \boldsymbol{y}) = 0 \quad \rho\text{-a.s. in } \Gamma$$

(+ initial and boundary conditions) for elliptic differential operator $A(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{y} \in \Gamma \subset \mathbb{R}^d$.

$$\boxed{u^{h,l,\delta}(\boldsymbol{x}, t, \boldsymbol{y}) := \sum_{\boldsymbol{z} \in \mathcal{Z}^l} u^{h,\delta}(\boldsymbol{x}, t; \boldsymbol{z}) L_{\boldsymbol{z}}^l(\boldsymbol{y})}$$

Construct an approximation for use in

- Forward, Inverse Uncertainty Quantification
- Sensitivity Analysis
- Optimal Control ...

Problem Discretisation (I)

1 **Spatial discretisation**: Galerkin FEM

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^h(t,\boldsymbol{y}) + A^h(\boldsymbol{y})\boldsymbol{u}^h(t,\boldsymbol{y}) = \boldsymbol{f}^h(t,\boldsymbol{y})$$

[1] Smolyak. 1963; Barthelmann, Novak, and Ritter. 2000; Babuška, Nobile, and Tempone. 2007.

Problem Discretisation (I)

1 **Spatial discretisation**: Galerkin FEM

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^h(t,\boldsymbol{y}) + A^h(\boldsymbol{y})\boldsymbol{u}^h(t,\boldsymbol{y}) = \boldsymbol{f}^h(t,\boldsymbol{y})$$

2 **Parametric discretisation**: **Sparse Grid Interpolation**[1]

[1] Smolyak. 1963; Barthelmann, Novak, and Ritter. 2000; Babuška, Nobile, and Tempone. 2007.

Problem Discretisation (I)

1 **Spatial discretisation**: Galerkin FEM

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^h(t,\boldsymbol{y}) + A^h(\boldsymbol{y})\boldsymbol{u}^h(t,\boldsymbol{y}) = \boldsymbol{f}^h(t,\boldsymbol{y})$$

2 **Parametric discretisation**: **Sparse Grid Interpolation**[1]

For multi-index set $I = \{\underline{\alpha}_1, \underline{\alpha}_2, ...\}$

$$\begin{aligned} u^{h,I}(\boldsymbol{x},t,\boldsymbol{y}) :=& \mathcal{I}^I[u^h(\boldsymbol{x},t,\boldsymbol{y})] \\ =& \sum_{\underline{\alpha}\in I}\bigotimes_{i=1}^{d}\Delta^{\alpha_i}[u^h(\boldsymbol{x},t,\boldsymbol{y})] \\ =& \sum_{\boldsymbol{z}\in\mathcal{Z}^I} u^h(\boldsymbol{x},t;\boldsymbol{z})L^I_{\boldsymbol{z}}(\boldsymbol{y}) \end{aligned}$$

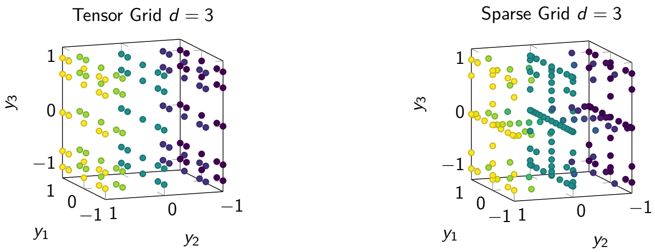
[1]Smolyak. 1963; Barthelmann, Novak, and Ritter. 2000; Babuška, Nobile, and Tempone. 2007.

Problem Discretisation (I)

1 **Spatial discretisation**: Galerkin FEM

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^h(t,\boldsymbol{y}) + A^h(\boldsymbol{y})\boldsymbol{u}^h(t,\boldsymbol{y}) = \boldsymbol{f}^h(t,\boldsymbol{y})$$

2 **Parametric discretisation**: **Sparse Grid Interpolation**[1]



[1] Smolyak. 1963; Barthelmann, Novak, and Ritter. 2000; Babuška, Nobile, and Tempone. 2007.

Problem Discretisation (II)

3 **ADAPTIVE** Timestepping: Implicit method with **local** error control.[2]

- ▷ Selects time steps Δt_1, Δt_2, $\Delta t_3, \ldots$ such that

$$\| \boldsymbol{u}^h(t + \Delta t_k, \boldsymbol{z}; \boldsymbol{u}_k^{h,\delta}) - \boldsymbol{u}_{k+1}^{h,\delta}(\boldsymbol{z}; \boldsymbol{u}_k^{h,\delta}) \|_{L^2(D)} \leq \delta$$

  where $\boldsymbol{u}_k^{h,\delta} \approx \boldsymbol{u}^h(t_k)$.
- ▷ Extend to continuous time $u^{h,\delta}(\boldsymbol{x}, t; \boldsymbol{z}) \approx u^h(\boldsymbol{x}, t; \boldsymbol{z})$.

[2] Gresho, Griffiths, and Silvester. 2008.

Problem Discretisation (II)

3 **ADAPTIVE** Timestepping: Implicit method with **local** error control.[2]

- Selects time steps Δt_1, Δt_2, Δt_3, ... such that

$$\|\boldsymbol{u}^h(t + \Delta t_k, \boldsymbol{z}; \boldsymbol{u}_k^{h,\delta}) - \boldsymbol{u}_{k+1}^{h,\delta}(\boldsymbol{z}; \boldsymbol{u}_k^{h,\delta})\|_{L^2(D)} \leq \delta$$

  where $\boldsymbol{u}_k^{h,\delta} \approx \boldsymbol{u}^h(t_k)$.
- Extend to continuous time $u^{h,\delta}(\boldsymbol{x}, t; \boldsymbol{z}) \approx u^h(\boldsymbol{x}, t; \boldsymbol{z})$.

- **Final approximation:**

$$u^{h,l,\delta}(\boldsymbol{x}, t, \boldsymbol{y}) := \sum_{\boldsymbol{z} \in \mathcal{Z}^l} u^{h,\delta}(\boldsymbol{x}, t; \boldsymbol{z}) L_{\boldsymbol{z}}^l(\boldsymbol{y})$$

[2] Gresho, Griffiths, and Silvester. 2008.

Toy Example

Consider[3] $u : D \times [0, T] \times \Gamma \to \mathbb{R}$ such that ρ-a.s.

$$\frac{\partial u(\boldsymbol{x}, t, \boldsymbol{y})}{\partial t} - \epsilon \nabla^2 u(\boldsymbol{x}, t, \boldsymbol{y}) + \boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla u(\boldsymbol{x}, t, \boldsymbol{y}) = 0$$

$\Gamma = [-1, 1]^d \subset \mathbb{R}^d$. $\nabla \cdot \boldsymbol{w} \equiv 0$, $\epsilon = 0.1$.

[3] Elman, Silvester, and Wathen. 2014.

Toy Example

Consider[3] $u : D \times [0, T] \times \Gamma \to \mathbb{R}$ such that ρ-a.s.

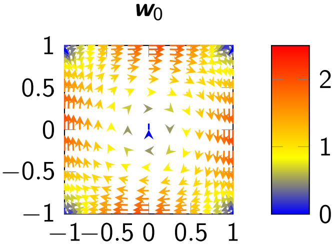
$$\frac{\partial u(\boldsymbol{x}, t, \boldsymbol{y})}{\partial t} - \epsilon \nabla^2 u(\boldsymbol{x}, t, \boldsymbol{y}) + \boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla u(\boldsymbol{x}, t, \boldsymbol{y}) = 0$$

$\Gamma = [-1, 1]^d \subset \mathbb{R}^d$. $\nabla \cdot \boldsymbol{w} \equiv 0$, $\epsilon = 0.1$.

- Wind field
  $\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{w}_0(\boldsymbol{x}) + \sum_{i=1}^{d} \lambda_i y_i \boldsymbol{w}_i(\boldsymbol{x})$.

- Hot wall BC
  $u(\boldsymbol{x}, t, \boldsymbol{y}) = (1 - x_2^4)(1 - \exp(-t/\tau))$
  for $x_1 = 1$, zero elsewhere.

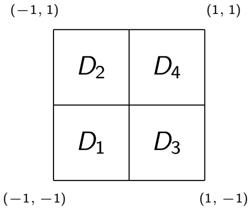
- Initial Condition
  $u(\boldsymbol{x}, 0, \boldsymbol{y}) = 0$ for all $(\boldsymbol{x}, \boldsymbol{y}) \in D \times \Gamma$.



[3] Elman, Silvester, and Wathen. 2014.

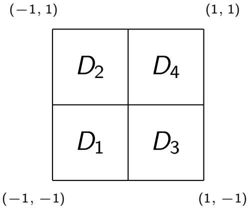
Toy Example: $d = 4$

- $\boldsymbol{w}_i(\boldsymbol{x}) := \boldsymbol{w}_0(2(x_1 - a_i), 2(x_2 - b_i))$ for $\boldsymbol{x} \in D_i$,
- $\boldsymbol{w}_i(\boldsymbol{x}) = 0$ otherwise,
- $\lambda_i = 0.5$ for $i = 1, 2, 3, 4$.

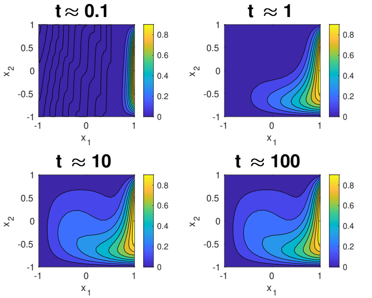


Toy Example: $d = 4$

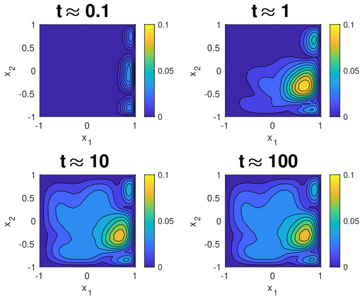
- $\boldsymbol{w}_i(\boldsymbol{x}) := \boldsymbol{w}_0(2(x_1 - a_i), 2(x_2 - b_i))$ for $\boldsymbol{x} \in D_i$,
- $\boldsymbol{w}_i(\boldsymbol{x}) = 0$ otherwise,
- $\lambda_i = 0.5$ for $i = 1, 2, 3, 4$.



- $I = \{\|\underline{\alpha}\|_1 \leq 4 + 5\}$, $\delta = 10^{-8}$.



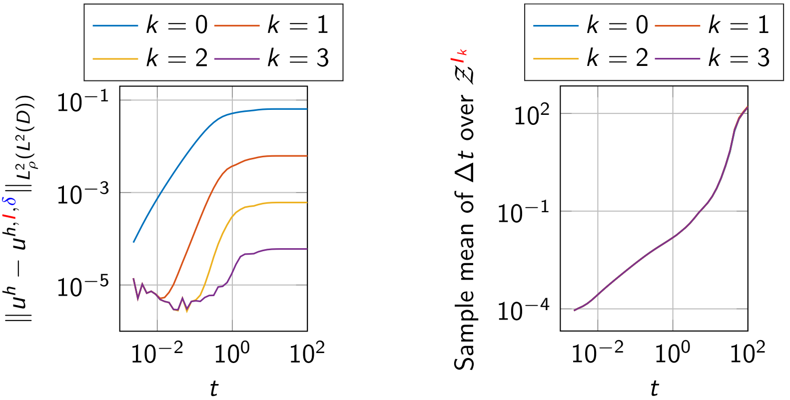
$\mathbb{E}[u^{h,I,\delta}(\boldsymbol{x}, t, \cdot)]$ for $t \approx 0.1, 1, 10, 100$



$\mathrm{stddev}[u^{h,I,\delta}(\boldsymbol{x}, t, \cdot)]$ for $t \approx 0.1, 1, 10, 100$

Approximation Error and Timestep Evolution

Approximation of u^h with Smolyak grids with $I_k = \{\|\alpha\|_1 \leq d + k\}$ and timestepping LE tolerance $\delta = \mathcal{O}(10^{-8})$ (GE tol $\mathcal{O}(10^{-5})$).



Small t	**LoFi** parameter approximation	**HiFi** time-stepping
Large t	**HiFi** parameter approximation	**LoFi** time-stepping

Error Estimation (I)

Consider error as function of time $e(t) := \left\| u(\cdot, t, \cdot) - u^{h,I,\delta}(\cdot, t, \cdot) \right\|_{L^2_\rho(L^2(D))}$

$$e = \| \underbrace{u - u^h}_{spatial} + \underbrace{u^h - u^{h,I}}_{interpolation} + \underbrace{u^{h,I} - u^{h,I,\delta}}_{timestepping} \|_{L^2_\rho(L^2(D))}$$

Error Estimation (II)

- **Interpolation Error**: Choose $\mathcal{I}^{I^*} := \mathcal{I}^{I \cup \mathcal{R}_I}$. **Saturation assumption** results in constant C such that

$$\|u^h - u^{h,I}\|_{L^2_\rho(L^2(D))} \leq C \|u^{h,I^*} - u^{h,I}\|_{L^2_\rho(L^2(D))}$$

Error Estimation (II)

- **Interpolation Error**: Choose $\mathcal{I}^{I^*} := \mathcal{I}^{I \cup \mathcal{R}_I}$. **Saturation assumption** results in constant C such that

$$
\begin{aligned}
\|u^h - u^{h,I}\|_{L^2_\rho(L^2(D))} &\le C\|u^{h,I^*} - u^{h,I}\|_{L^2_\rho(L^2(D))} \\
&\le \underbrace{C\|u^{h,I^*,\delta} - u^{h,I,\delta}\|_{L^2_\rho(L^2(D))}}_{\pi_{interp}(t)} \\
&\quad \underbrace{+C\Bigg(\sum_{z \in \mathcal{Z}^{I^*} \setminus \mathcal{Z}^I} \|e^\delta(\cdot, t; z)\|_{L^2(D)} \|L_z^{I^*}\|_{L^2_\rho(\Gamma)} + \sum_{z \in \mathcal{Z}^I} \|e^\delta(\cdot, t; z)\|_{L^2(D)} \|L_z^{I^*} - L_z^I\|_{L^2_\rho(\Gamma)}\Bigg)}_{\pi_{corr}(t)}
\end{aligned}
$$

Error Estimation (II)

- **Interpolation Error**: Choose $\mathcal{I}^{\mathbf{l}^*} := \mathcal{I}^{\mathbf{l} \cup \mathcal{R}_{\mathbf{l}}}$. **Saturation assumption** results in constant C such that

$$\begin{aligned}
\| u^h - u^{h,\mathbf{l}} \|_{L^2_\rho(L^2(D))} \leq & C \| u^{h,\mathbf{l}^*} - u^{h,\mathbf{l}} \|_{L^2_\rho(L^2(D))} \\
\leq & \underbrace{C \| u^{h,\mathbf{l}^*,\delta} - u^{h,\mathbf{l},\delta} \|_{L^2_\rho(L^2(D))}}_{\pi_{interp}(t)} \\
& \underbrace{+ C \Bigg(\sum_{\mathbf{z} \in \mathcal{Z}^{\mathbf{l}^*} \setminus \mathcal{Z}^{\mathbf{l}}} \| e^\delta(\cdot, t; \mathbf{z}) \|_{L^2(D)} \| L_{\mathbf{z}}^{\mathbf{l}^*} \|_{L^2_\rho(\Gamma)} + \sum_{\mathbf{z} \in \mathcal{Z}^{\mathbf{l}}} \| e^\delta(\cdot, t; \mathbf{z}) \|_{L^2(D)} \| L_{\mathbf{z}}^{\mathbf{l}^*} - L_{\mathbf{z}}^{\mathbf{l}} \|_{L^2_\rho(\Gamma)} \Bigg)}_{\pi_{corr}(t)}
\end{aligned}$$

- Split as
$\pi_{\text{interp}}(t) \leq \sum_{\underline{\alpha} \in \mathcal{R}_{\mathbf{l}}} \left\| \bigotimes_{j=1}^d \Delta^{\alpha_j} [u^{h,\delta}(\cdot, t; \mathbf{y})] \right\|_{L^2_\rho(L^2(D))} = \sum_{\underline{\alpha} \in \mathcal{R}_{\mathbf{l}}} \pi_{\text{interp},\underline{\alpha}}(t)$

Error Estimation (III)

▷ **Timestepping Error**:

$$\|u^{h,l} - u^{h,l,\delta}\|_{L^2_\rho(L^2(D))} \leq \sum_{z \in \mathcal{Z}^l} \underbrace{\|e^{\delta}(\cdot, t; \mathbf{z})\|_{L^2(D)}}_{\text{Global timestepping error}} \|L^l_{\mathbf{z}}(\mathbf{y})\|_{L^2_\rho(\Gamma)} =: \pi_{ts}(t).$$

[4]Skeel. 1986.
[5]Shampine. 1994.

Error Estimation (III)

- **Timestepping Error**:

$$\|u^{h,l} - u^{h,l,\delta}\|_{L^2_\rho(L^2(D))} \leq \sum_{z \in \mathcal{Z}^l} \underbrace{\|e^{\delta}(\cdot, t; z)\|_{L^2(D)}}_{\text{Global timestepping error}} \|L^l_z(y)\|_{L^2_\rho(\Gamma)} =: \pi_{ts}(t).$$

- Need strategy to compute an estimate $\pi_{ge}(t; z) \approx \|e^{\delta}(\cdot, t; z)\|_{L^2(D)}$.
- Use two different **local error** tolerances[4] and scaling argument[5]

$$\pi_{ge}(t) = \left(\frac{\delta}{\delta_0}\right)^{p/p+1} \pi^{\delta_0}_{ge}(t).$$

- Using π_{ge}, we have a computable bound

$$\pi := \pi_{\text{interp}} + \pi_{\text{corr}} + \pi_{\text{ts}}.$$

[4]Skeel. 1986.
[5]Shampine. 1994.

Adaptive SC Algorithm

$$\cdots \to \textit{SOLVE} \to \textit{ESTIMATE} \to \textit{MARK} \to \textit{REFINE} \to \cdots$$

1 Initialise time $t = 0$, $\mathcal{Z} = \{\mathbf{0}\}$.

2 Solve linear systems associated with each collocation point $\boldsymbol{z} \in \mathcal{Z}^*$ to time $t + \Delta t$.

3 Estimate $\pi_{\text{interp}}, \pi_{\text{corr}}, \pi_{\text{ts}}, \{\pi_{\text{interp},\underline{\alpha}}\}_{\underline{\alpha} \in \mathcal{R}_I}$

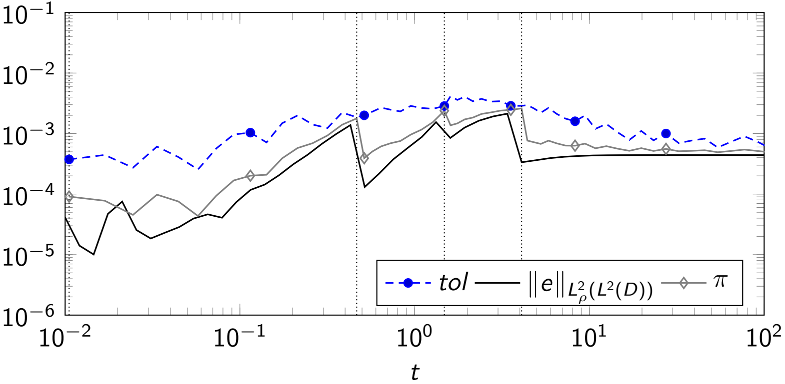
4*a* If $\pi_{\text{interp}} \geq \textit{tol} \propto \pi_{\text{corr}}$, then mark $\mathcal{J} \subset \mathcal{R}_I$ and refine I.

4*b* Else accept timestep $t \leftarrow t + \Delta t$.

5 Return to step 2 and repeat until $t = T$.

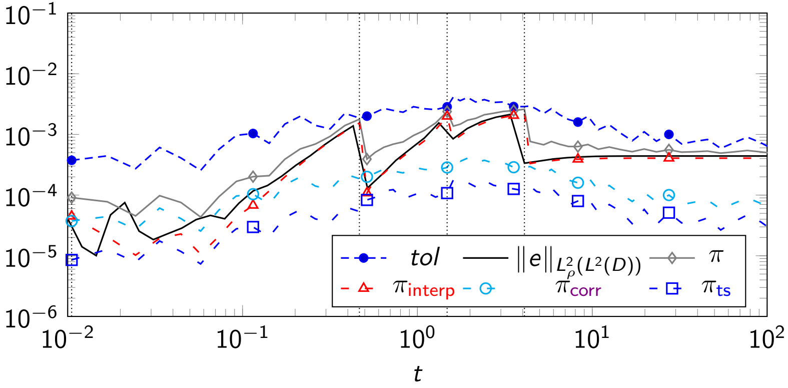
Adaptive Approximation $d = 4$

$\pi, \pi_{\text{interp}}, \pi_{\text{corr}}, \pi_{\text{ts}}$ for the parametric $d = 4$ double glazing problem (vertical dotted lines denote parametric refinement)



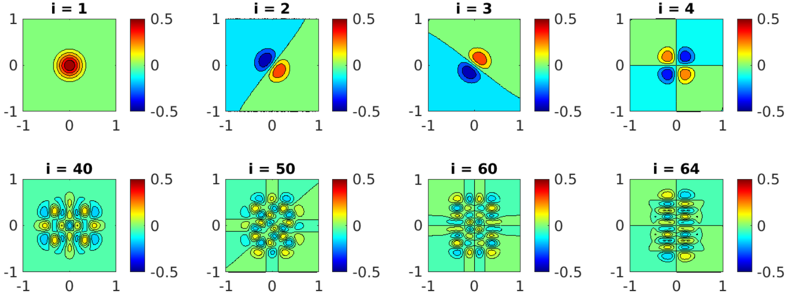
Adaptive Approximation $d = 4$

$\pi, \pi_{\text{interp}}, \pi_{\text{corr}}, \pi_{\text{ts}}$ for the parametric $d = 4$ double glazing problem
(vertical dotted lines denote parametric refinement)



Extension to higher dimensions ($d = 64$)

$$w(\boldsymbol{x}, \boldsymbol{y}) := w_0(\boldsymbol{x}) + \sum_{i=1}^{64} (\nabla \times \phi_i(\boldsymbol{x})) y_i$$

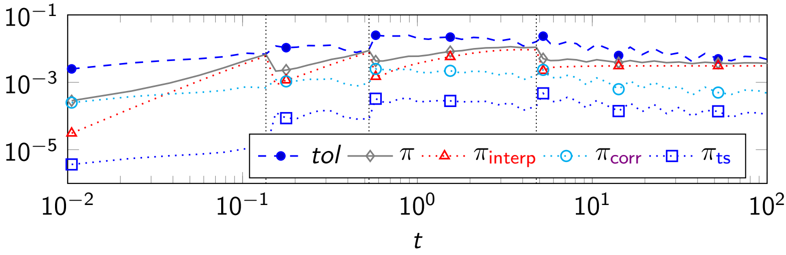


Stream functions of perturbations

Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs.* URL: https://arxiv.org/abs/2210.03389

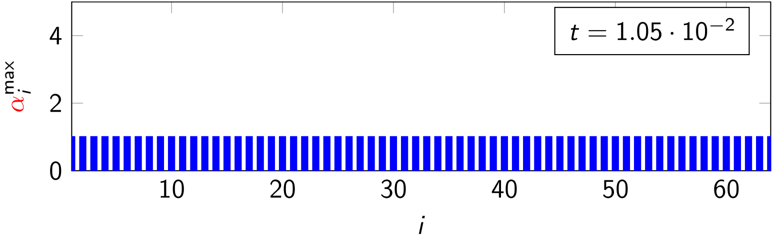
π, π_{interp}, π_{corr}, π_{ts} for the parametric $d = 64$ double glazing problem



Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs.* URL: https://arxiv.org/abs/2210.03389

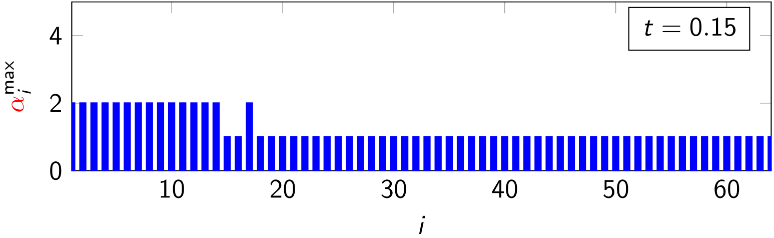
Maximum interpolant level $\alpha_i^{\max} := \max_{\underline{\alpha} \in I} \alpha_i$ for each parameter dimension i at snapshots in time



Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs.* URL: https://arxiv.org/abs/2210.03389

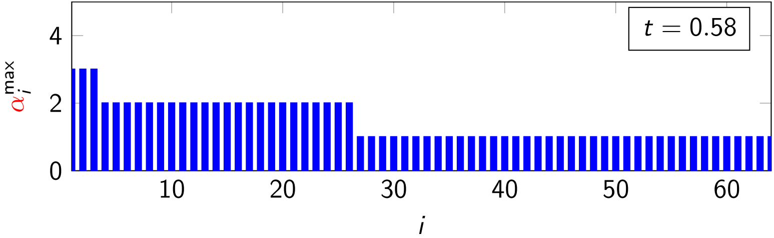
Maximum interpolant level $\alpha_i^{\max} := \max_{\underline{\alpha} \in I} \alpha_i$ for each parameter dimension i at snapshots in time



Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs.* URL: https://arxiv.org/abs/2210.03389

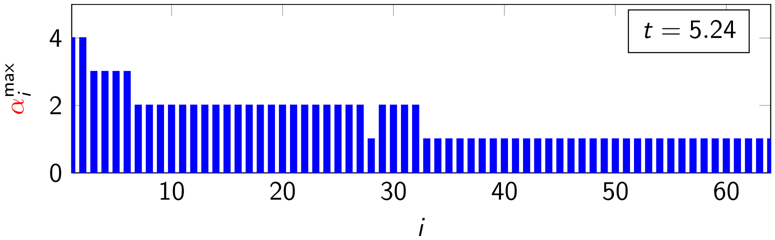
Maximum interpolant level $\alpha_i^{\max} := \max_{\underline{\alpha} \in I} \alpha_i$ for each parameter dimension i at snapshots in time



Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs.* URL: https://arxiv.org/abs/2210.03389

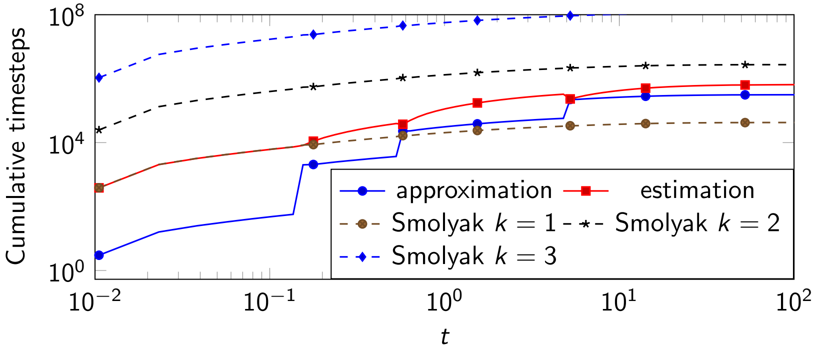
Maximum interpolant level $\alpha_i^{\max} := \max_{\underline{\alpha} \in I} \alpha_i$ for each parameter dimension i at snapshots in time



Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs*. URL: https://arxiv.org/abs/2210.03389

Computational cost (total number of timesteps)



Numerical Results ($d = 64$)

Benjamin M. Kent et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs*. URL: https://arxiv.org/abs/2210.03389

Adaptive approximation for parametric problems is **ESSENTIAL!**

Babuška, Ivo, Fabio Nobile, and Raúl Tempone (2007). "A Stochastic Collocation Method for Elliptic Partial Differential Equations with Random Input Data". In: *SIAM Journal on Numerical Analysis* 45.3, pp. 1005–1034. DOI: 10.1137/050645142. eprint: https://doi.org/10.1137/050645142. URL: https://doi.org/10.1137/050645142.

Barthelmann, Volker, Erich Novak, and Klaus Ritter (Mar. 2000). "High dimensional polynomial interpolation on sparse grids". In: *Advances in Computational Mathematics* 12.4, pp. 273–288. ISSN: 1572-9044. DOI: 10.1023/A:1018977404843.

Black, Fischer and Myron Scholes (1973). "The pricing of options and corporate liabilities". In: *Journal of political economy* 81.3, pp. 637–654.

Elman, Howard, David J. Silvester, and Andy Wathen (2014). *Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics*. Second. Oxford, UK: Oxford University Press. ISBN: 9780191523786. DOI: 10.1093/acprof:oso/9780199678792.001.0001.

- Gresho, Philip M., David F. Griffiths, and David J. Silvester (2008). "Adaptive Time-Stepping for Incompressible Flow Part I: Scalar Advection-Diffusion". In: *SIAM Journal on Scientific Computing* 30.4, pp. 2018–2054. DOI: 10.1137/070688018.
- Kent, Benjamin M. et al. (2022). *Efficient Adaptive Stochastic Collocation Strategies for Advection-Diffusion Problems with Uncertain Inputs*. URL: https://arxiv.org/abs/2210.03389.
- Markowich, P.A., C.A. Ringhofer, and C. Schmeiser (1990). *Semiconductor Equations*. Springer. ISBN: 9780387821573.
- Quarteroni, Alfio, Alessandro Veneziani, and Paolo Zunino (2002). "Mathematical and Numerical Modeling of Solute Dynamics in Blood Flow and Arterial Walls". In: *SIAM Journal on Numerical Analysis* 39.5, pp. 1488–1511. DOI: 10.1137/S0036142900369714. eprint: https://doi.org/10.1137/S0036142900369714. URL: https://doi.org/10.1137/S0036142900369714.
- Shampine, Lawrence F. (1994). *Numerical solution of ordinary differential equations*. English. Chapman & Hall mathematics. New York, NY; London: Chapman & Hall. ISBN: ISBN: 0412051516.

Skeel, Robert D. (Jan. 1986). "Thirteen ways to estimate global error". In: *Numerische Mathematik* 48.1, pp. 1–20. ISSN: 0029-599X. DOI: 10.1007/BF01389440. URL: http://link.springer.com/10.1007/BF01389440.

Smolyak, S A (1963). "Quadrature and interpolation formulae on tensor products of certain function classes". In: *Soviet Math. Dokl.* 4.5, pp. 240–243.

Stockie, John M. (Jan. 2011). "The Mathematics of Atmospheric Dispersion Modeling". In: *SIAM Review* 53.2, pp. 349–372. DOI: 10.1137/10080991x. URL: https://doi.org/10.1137/10080991x.

Stream function based expansion $d = 64$

- Use $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y}) = \nabla \times \psi(\boldsymbol{x},\boldsymbol{y})$ for $\psi(\boldsymbol{x},\boldsymbol{y}) = \psi_0(\boldsymbol{x}) + \sum_{i=1}^{d} \sqrt{\lambda_i}\psi_i(\boldsymbol{x})y_i$.
- $\boldsymbol{w}_0 = \nabla \times \left(-(1-x_1^2)(1-x_2^2)\right)$
- Approximate eigenpairs (λ_i, ψ_i) of $C(\boldsymbol{x}_1,\boldsymbol{x}_2) = \prod_{i,j=1}^{2}(1-x_{i,j}^2)\sigma_0^2 \exp\left(-\frac{\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2^2}{L}\right)$
- Choose $\sigma_0^2 = 5,\ L = 1,\ d = 64.$

π, π_{interp}, π_{corr}, π_{ts} for the parametric $d = 64$ double glazing problem